B. Math. Hons. IInd year
Second Semester Backpaper Examination 2025
Rings and Modules
Instructor - B. Sury
Maximum Marks 50
Answer 5 questions INCLUDING Q1.

Q 1. Determine if the following statements are true or false. If the statement is true, give brief reasoning; in case of falsity, provide a counter-example.
(a) The element $X^2 + 3X + 2 \in \mathbb{Z}[X]$ is not irreducible but becomes irreducible in some larger ring.
(b) In a Boolean ring, every ideal is principal.
(c) $\mathbb{R}[X, Y]/(X^2 + Y^2 - 1)$ is not a UFD.
(d) $\mathbb{C}[X, Y]/(X^2 + Y^2 - 1)$ is a PID.
(e) $\mathbb{Z}[\sqrt{8}]$ is a UFD.

Q 2. Let R be a commutative ring with unity. If M is a finitely generated R-module, and $M_1 \subseteq M_2 \subseteq M$ are submodules such that $M/M_1 \cong M/M_2$ as R-modules, prove that $M_1 = M_2$. If M is not finitely generated, then is the above necessarily true?

Q 3. Let M be a finitely generated R-module, and let $\phi : M \to M$ be a surjective R-module homomorphism. Prove that ϕ must be an isomorphism.

Q 4.
(a) For a commutative ring A with unity, show that an ideal I is a direct summand of A if, and only if, I is principal, generated by an idempotent.
(b) For the ring $A = \mathbb{Z}[X]$, find an ideal I that is not free as an A-module.

Q 5. Let M be a free module of rank $n > 0$ over a PID. If N is a non-zero submodule of M, prove that N is also free, and has rank $\leq n$. Can the ranks be equal when $N \neq M$?

Q 6. Find the invariant factors of

$$\begin{pmatrix} X - 17 & 8 & 12 & -14 \\ -46 & X + 22 & 35 & -41 \\ 2 & -1 & X - 4 & 4 \\ -4 & 2 & 2 & X - 3 \end{pmatrix}.$$